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Recent progress on conditional randomness
Hayato Takahashi¹

The set of Hippocratic random sequences w.r.t. P is defined as the complement of the effective null sets w.r.t. P and denote it by \mathcal{R}^P . In particular if P is computable it is called Martin-Löf random sequences.

Lambalgen's theorem (1987) [9] says that a pair of sequences $(x^\infty, y^\infty) \in \Omega^2$ is Martin-Löf (ML) random w.r.t. the product of uniform measures iff x^∞ is ML-random and y^∞ is ML-random relative to x^∞ , where Ω is the set of infinite binary sequences. In [10, 5, 6, 7], generalized Lambalgen's theorem is studied for computable correlated probabilities.

Let S be the set of finite binary strings and $\Delta(s) := \{sx^\infty | x^\infty \in \Omega\}$ for $s \in S$, where sx^∞ is the concatenation of s and x^∞ . Let $X = Y = \Omega$ and P be a computable probability on $X \times Y$. P_X and P_Y are marginal distribution on X and Y , respectively. In the following we write $P(x, y) := P(\Delta(x) \times \Delta(y))$ and $P(x|y) := P(\Delta(x) | \Delta(y))$ for $x, y \in S$.

Let \mathcal{R}^P be the set of ML-random points and $\mathcal{R}_{y^\infty}^P := \{x^\infty | (x^\infty, y^\infty) \in \mathcal{R}^P\}$. In [5, 6], it is shown that conditional probabilities exist for all random parameters, i.e.,

$$\forall x \in S, y^\infty \in \mathcal{R}^{P_Y} \quad P(x|y^\infty) := \lim_{y \rightarrow y^\infty} P(x|y) \text{ (the right-hand-side exist)}$$

and $P(\cdot|y^\infty)$ is a probability on (Ω, \mathcal{B}) for each $y^\infty \in \mathcal{R}^{P_Y}$.

Let $\mathcal{R}^{P(\cdot|y^\infty), y^\infty}$ be the set of Hippocratic random sequences w.r.t. $P(\cdot|y^\infty)$ with oracle y^∞ .

Theorem 1 ([5, 6, 7]) *Let P be a computable probability on $X \times Y$. Then*

$$\mathcal{R}_{y^\infty}^P \supseteq \mathcal{R}^{P(\cdot|y^\infty), y^\infty} \text{ for all } y^\infty \in \mathcal{R}^{P_Y}. \quad (1)$$

Fix $y^\infty \in \mathcal{R}^{P_Y}$ and suppose that $P(\cdot|y^\infty)$ is computable with oracle y^∞ . Then

$$\mathcal{R}_{y^\infty}^P = \mathcal{R}^{P(\cdot|y^\infty), y^\infty}. \quad (2)$$

¹ hayato.takahashi@ieee.org

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It is known that there is a non-computable conditional probabilities [4] and in [2] Bauwens showed an example that violates the equality in (2) when the conditional probability is not computable with oracle y^∞ . In [8], an example that for all y^∞ , the conditional probabilities are not computable with oracle y^∞ and (2) holds. A survey on the randomness for conditional probabilities is shown in [1].

Next we study mutually singular conditional probabilities. In [3], Hanssen showed that for Bernoulli model $P(\cdot|\theta)$,

$$\mathcal{R}^{P(\cdot|\theta)} = \mathcal{R}^{P(\cdot|\theta),\theta} \text{ for all } \theta. \quad (3)$$

We generalize Hanssen's theorem (3) for mutually singular conditional probabilities. In [5, 7], equivalent conditions for mutually singular conditional probabilities are shown.

Theorem 2 ([5, 7]) *Let P be a computable probability on $X \times Y$, where $X = Y = \Omega$. The following six statements are equivalent:*

- (1) $P(\cdot|y) \perp P(\cdot|z)$ if $\Delta(y) \cap \Delta(z) = \emptyset$ for $y, z \in S$.
- (2) $\mathcal{R}^{P(\cdot|y)} \cap \mathcal{R}^{P(\cdot|z)} = \emptyset$ if $\Delta(y) \cap \Delta(z) = \emptyset$ for $y, z \in S$.
- (3) $P_{Y|X}(\cdot|x)$ converges weakly to I_{y^∞} as $x \rightarrow x^\infty$ for $(x^\infty, y^\infty) \in \mathcal{R}^P$, where I_{y^∞} is the probability that has probability of 1 at y^∞ .
- (4) $\mathcal{R}_{y^\infty}^P \cap \mathcal{R}_{z^\infty}^P = \emptyset$ if $y^\infty \neq z^\infty$.
- (5) There exists $f : X \rightarrow Y$ such that $f(x^\infty) = y^\infty$ for $(x^\infty, y^\infty) \in \mathcal{R}^P$.
- (6) There exists $f : X \rightarrow Y$ and $Y' \subset Y$ such that $P_Y(Y') = 1$ and $f = y^\infty$, $P(\cdot; y^\infty) - \text{a.s.}$ for $y^\infty \in Y'$.

Generalized form of Hanssen's theorem (3) is as follows.

Theorem 3 *Let P be a computable probability on $X \times Y$, where $X = Y = \Omega$. Under one of the condition of Theorem 2, we have*

$$\mathcal{R}_{y^\infty}^P \supseteq \mathcal{R}^{P(\cdot|y^\infty)} \text{ for all } y^\infty \in \mathcal{R}^{P_Y}.$$

Fix $y^\infty \in \mathcal{R}^{P_Y}$ and suppose that $P(\cdot|y^\infty)$ is computable with oracle y^∞ . Then

$$\mathcal{R}_{y^\infty}^P = \mathcal{R}^{P(\cdot|y^\infty)} = \mathcal{R}^{P(\cdot|y^\infty), y^\infty}.$$

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